T.C. Dokuz Eylül Üniversitesi İktisadî ve İdarî Bilimler Fakültesi Dergisi, Cilt: 1, No: 1, Yıl: 1986

DERLEME

# SALES ESTIMATIONS AND PROFIT ANALYSES WHEN DEMAND HAS A NONLINEAR, EXPONENTIAL TREND AND NORMALLY DISTRIBUTED

**Sedat AKALIN(*)**

**SUMMARY**

It is clear enough that, to be able to continue their activities, the firms depend on their sales to exceed all the expenses they have to face.

The estimating of selling with a small error ratio is light up on the business decisions because of the future selling is not exact defined. After the estimating of quantity of selling; prices, costs could be estimate and calculate for the same future time period. The profit can be found by subtracting total cost from total income in the same period.

The estimating of selling are depends on the demand functions which are found from the market researches. For this reason there are many case of demand functions such as exponential, normal, curve linear, logaritmie, poison, uniform distributions etc. But we will only be interested in the first three cases.

---

(*) Prof. Dr., D.E.Ü.İ.İ.B.F., Ekonometri Bölümü

**I. Estimating Sales and Profits when Demand has a curvilinear trend**

Demand, as everbody knows, is the maximum amount of commodity or service that consumers (i.e. households) wish to purchase off the market at all possible alternative prices, in a certain period of time, other factors held constant.

Functional relation between price and quantity demanded may be stated as

$$q = f(p) \qquad (1)$$

or

$$f(q,p) = 0 \qquad (2)$$

where

q is the number of units demanded,

p is the market price of the commodity subject to study.

Leaving aside the other factors effecting demand, the simplest one of the nonlinear demand functions is in the shape of

$$q_h = a + bp + cp^2 \qquad (3)$$

and the ones having higher powers are also analized same way, so they are not included here.

When the principle of «the sum of squares of vertical devlations from regression or trend curve is minimum» applied to Eq. (3), for determining the quantities demanded which change inversly with price, and putting

$$D = \Sigma (q - q_h)^2$$

the following may be written:

$$1^\circ \quad \frac{\partial D}{\partial a} = \frac{\partial \Sigma (q-a-bp-cp^2)^2}{\partial a} = -2\Sigma(q-a-bp-cp^2) = 0$$

$$2^\circ \quad \frac{\partial D}{\partial b} = \frac{\partial \Sigma (q-a-bp-cp^2)^2}{\partial b} = -2\Sigma p(q-a-bp-cp^2) = 0$$

$$3^\circ \quad \frac{\partial D}{\partial c} = \frac{\partial \Sigma (q-a-bp-cp^2)}{\partial c} = -2\Sigma p^2(q-a-bp-cp^2) = 0$$

and from 1°, 2°, 3° normal equations are obtained as



$$\text{I} \quad \sum q = na + b\sum p + c\sum p^2$$

$$\text{II} \quad \sum qp = a\sum p + b\sum p^2 + c\sum p^3$$

$$\text{III} \quad \sum qp^2 = a\sum p^2 + b\sum p^3 + c\sum p^4$$

When the deviations from the average price, viz. $p_s = p - \overline{p}$ are taken into consideration, the above written normal equations may be shortened, in turn, as follows:

$$\text{I}' \quad \sum q = na + c\sum p_s^2$$

$$\text{II}' \quad \sum qp_s = b\sum p_s^2$$

$$\text{III}' \quad \sum qp_s^2 = a\sum p_s^2 + c\sum p_s^4$$

and parameters are determined in a shorter time and easily:

$$a = \frac{\begin{vmatrix} \sum q & \sum p_s^2 \\ \sum qp_s^2 & \sum p_s^4 \end{vmatrix}}{\begin{vmatrix} n & \sum p_s^2 \\ \sum p_s^2 & \sum p_s^4 \end{vmatrix}} = \frac{(\sum q)(\sum p_s^4) - (\sum qp_s^2)(\sum p_s^2)}{n\sum p_s^4 - (\sum p_s^2)^2}$$

$$b = \frac{\sum qp_s}{\sum p_s^2} \quad ; \quad c = \frac{\sum q - na}{\sum p_s^2}$$

1) Parameters of demand function thus found are substituted into Eq. (3) to get Eq (4):

$$q_n = a + bp_s + cp_s^2 \qquad (4)$$

2) From the price trend, estimated price of the required year may be obtained by Eq.(5) below

$$p_n = \alpha + \beta t_s \qquad (5)$$

where,

$t_s$ = deviations from middle-year, $t_s$ : —,—,...,—2,—1,0, 1,,2,...

$\alpha$ = average price, $\overline{p}$

$\beta$ = average annual change of price; i.e., slope of the price trend

3) Total revenue, $TR = q_n p_n \qquad (6)$

Putting above found $q_n$ into total cost function,

$$TC = b_0 + b_1 q + b_2 q^2 \qquad (7)$$

total cost of production is reached.

4) Since profit is the difference between revenue and expences,

$$K = TR - TC \qquad (8)$$

$$= q_n p_n - [b_0 + b_1 q_n + b_2 q_n^2] \qquad (9)$$

it is calculated thru Eq. (9),
where,

$$b_0 = \frac{(\sum TC)(\sum q_s^4) - (\sum q_s^2)(\sum q_s^2 \cdot TC)}{n \sum q_s^4 - (\sum q_s^2)^2}$$

$$b_1 = \frac{\sum (q_s \cdot TC)}{\sum q_s^2}$$

$$b_2 = \frac{\sum (TC) - n b_0}{\sum q_s^2}$$

$$q_s = q_i - \overline{q}$$



**Numerical Example**

A businessman, between the years 1977 - 1985, has faced below written prices, demands, and total costs. Estimation of 1988 sales and profits are desired.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Years** | : | 1977 | 1978 | 1979 | 1980 | 1981 | 1982 | 1983 | 1984 | 1985 |
| **Demands** | : | 253 | 246 | 240 | 230 | 220 | 215 | 214 | 214 | 220 |
| **Prices** | : | 12 | 18 | 25 | 36 | 49 | 63 | 72 | 84 | 100 |
| **Costs** | : | 1600 | 2600 | 3800 | 5300 | 6800 | 8500 | 9400 | 11000 | 14000 |

**SOLUTION**

First of all, trends of, 1. price, 2. demand, 3. total costs are to be fixed.

1. Parameters of $\alpha$ and $\beta$ will be computed from Table 1. :

| | $t_i$ | $p_i$ | $t_{si}=t_i-\bar{t}$ | $p_i \cdot t_{si}$ | $t_{si}^2$ |
|---|---|---|---|---|---|
| | 1977 | 12 | -4 | -48 | 16 |
| | 1978 | 18 | -3 | -54 | 9 |
| | 1979 | 25 | -2 | -50 | 4 |
| | 1980 | 36 | -1 | -36 | 1 |
| $\bar{t}$ = | 1981 | 49 | 0 | 0 | 0 |
| | 1982 | 63 | 1 | 63 | 1 |
| | 1983 | 72 | 2 | 144 | 2 |
| | 1984 | 84 | 3 | 252 | 9 |
| | 1985 | 100 | 4 | 400 | 16 |
| totals: | | 459 | Ø | 671 | 60 |

$\therefore \ \alpha = \bar{p}_i = 459/9 = 51$ m.u.

$$\beta = (\sum p_i t_{s_i}) / (\sum t_{s_i}^2) = 671/60 \triangleq 11{,}18$$

This way, $p_n = 51 + 11.18(t_s)$

holds. Since 1988 is 7 years later than the middle year 1981, $t_s = 7$ and price estimation of given year is:

$$\dot{p}_{1988} = 51 + 11{,}18(7) \doteq 129 \text{ mu.u. (i.e., monetary unit)}$$

2. Table 2 is arranged for establishing the demand trend the enterpriser has to face:

Table 2

| $q_i$ | $p_i$ | $p_s$ | $p_s^2$ | $p_s^4$ | $uq_s$ | $qp_s^2$ |
|---|---|---|---|---|---|---|
| 253 | 12 | —39 | 1521 | 2313441 | —9867 | 384813 |
| 246 | 18 | —33 | 1089 | 1185921 | —8118 | 267984 |
| 240 | 25 | —26 | 676 | 456976 | —6240 | 162240 |
| 230 | 36 | —15 | 225 | 50625 | —3450 | 51750 |
| 220 | 49 | — 2 | 4 | 16 | — 440 | 880 |
| 215 | 63 | 12 | 144 | 20736 | 2580 | 30960 |
| 214 | 72 | 21 | 441 | 194481 | 4494 | 94374 |
| 214 | 84 | 33 | 1089 | 1185921 | 7062 | 233046 |
| 220 | 100 | 49 | 2401 | 5764801 | 10780 | 528220 |
| Σ: 2052 | 459 | 0 | 7590 | 11172918 | —3199 | 1754177 |

$\bar{q} = 228$

$\bar{p} = 51$

$$a = \frac{2052(11172918) - 1754177(7590)}{9(1172918) - 57608100} \doteq 223{,}82$$

$$b = -3199/7590 = -0.42$$

$$c = \frac{2052 - 9(223{,}82)}{7590} = 0.0047$$

Then,

$$q_n = 223{,}82 - 0.42\, p_s + 0.0047 p_s^2$$

$$= 223.82 - 0.42(129-51) + 0.0047(129-51)^2$$

$$= 219{,}65 \text{ u.}$$

3. Total Cost trend will be computed thru Table 3 below.



| TC | $q_s$ | $q_s$.TC | $q_s^2$.TC | $q_s^2$ | $q_s^4$ |
|---|---|---|---|---|---|
| 1600 | 25 | 40000 | 1000000 | 625 | 390625 |
| 2600 | 18 | 46800 | 842400 | 324 | 104976 |
| 3800 | 12 | 45600 | 547200 | 144 | 20736 |
| 5300 | 2 | 10600 | 21200 | 4 | 16 |
| 6800 | —8 | —54400 | 435200 | 64 | 4096 |
| 8500 | —13 | —110500 | 1436500 | 169 | 28561 |
| 9400 | —14 | —131600 | 1842400 | 196 | 38416 |
| 11000 | —14 | —154000 | 2156000 | 196 | 38416 |
| 14000 | —8 | —112000 | 896000 | 64 | 4096 |
| Σ's: 63000 | Ø | —419500 | 9176900 | 1786 | 692938 |

$$b_0 = \frac{63000(692938) - 1786(9176900)}{9(692938) - (1786)^2} \doteq 8949$$

$$b_1 = \frac{-419500}{1786} \doteq -235 \ ; \quad b_2 = \frac{63000 - 9)8949)}{1786} \doteq -9{,}8$$

*** Total cost trend is,
$TC_n = 8949 - 235\,(q_n - 228) - 9{,}8\,(q_n - 228)^2$ and $q_n$ was 219,65, so

$$\dot{TC}_{1988} = 10223 \text{ m.u.}$$

Finally, profits, with its some zeroes neglected, is estimated as

$$\dot{K}_{1988} = 219{,}65(129) - 10223 = 18112 \text{ m.u.}$$

With this point estimation, proportion of error hasn't been considered, If it is desired, for instance, w/70% confidency of estimation, following way may be pursued:

I. Firstly, standart error of price regression,

II. Secondly, standard error of quantity regression must be computed:

I. Since, (9)

$$\hat{\sigma}_{p_t} = \sqrt{\frac{\sum p^2 - \alpha \sum p - \beta \sum p.t_s}{n-2}}$$

Table 4 is prepared for calculations:

Table 4

| | $p_i$ | $p_i^2$ | $t_{s_i}$ | $p_i t_{s_i}$ |
|---|---|---|---|---|
| | 12 | 144 | -4 | -48 |
| | 18 | 324 | -3 | -54 |
| | 25 | 625 | -2 | -50 |
| | 36 | 1296 | -1 | -36 |
| | 49 | 2401 | 0 | 0 |
| | 63 | 3969 | 1 | 63 |
| | 72 | 5184 | 2 | 144 |
| | 84 | 7056 | 3 | 252 |
| | 100 | 10000 | 4 | 400 |
| $\Sigma$'s: | 459 | 30999 | 0 | 671 |

$$\therefore \hat{\sigma}_{pt} = \sqrt{\frac{30999 - 51(459 - 11{,}18(671)}{7}} \doteq 3{,}55$$

Since $t_{.3;8} = 1{,}108$ price range of 1988 with 70% confidence is:

$$p_R = 129 \mp 1{,}108(3{,}55) \longrightarrow 125 \text{ upto } 133$$

This way, $q_1 = 223{,}82 - 0{,}42(125-51) + 0{,}0047(74^2) = 218{,}48$

$q_2 = 223{,}82 - 0{,}42(133-51) + 0{,}0047(82^2) = 221$

As to the range of profits,

$K_1 = 218{,}48(125) - [8949-235(218{,}48-228) -9{,}8(218{,}48-228)^2]$

$\doteq 17012$ m.u.



$K_2 = 221(133) - [8949-235(221-228) -9{,}8(221-228^2] \doteq 19279$ m.u.

That is to say, with the probability of 70%, profits of 1988 expected, at least, 17012 and 19279 m.u. at most.

[Note: $\hat{\sigma}_{qp_s^2} = \sqrt{\frac{\sum q^2 - a \sum q - b \sum qp_s - c \sum qp_s^2}{n-3}} \doteq 11{,}6$

would expand the range: $q_1 \doteq 208$; $q_2 \doteq 234$; $K_1 \doteq 16271$ and $K_2 \doteq 22526$.]

**2. The Case of Exponential Demand Function**

The general type of the functional relations between price and quantitiy demanded appears to be as follow:

$$q = ab^p \tag{I}$$

where,

- q = quantity demanded
- p = price of the commodity
- a = maximum quantity could be cleared off the market, positive
- b = price-parameter, positive and smaller than unity, $0 < b < 1$

Price and quantity are inversely related. For these exponential demand functions, logarithm has been taken first to reduce them into linear form, and then, above mentioned methods are applicable. Here, naturally, Napierian logarithms to the base $e = \lim_{n\to\infty} (1+1/n)^n \doteq 2{,}718\ldots$ preferred for simplicity:

$$\ln q = \ln a + p \ln b \tag{II}$$

1. $\sum \ln q = n \ln a + \ln b \sum p$

2. $\sum (p \ln q) = \ln a \sum p + \ln b \sum p^2$ (III)

Logarithms and then original values of the Parameters sought, may be established thru the relations stated below.

$$\ln a = \frac{\begin{vmatrix} \sum \ln q & \sum p \\ \sum (p \ln q) & \sum p^2 \end{vmatrix}}{\begin{vmatrix} n & \sum p \\ \sum p & \sum p^2 \end{vmatrix}} = \frac{(\sum \ln q)(\sum p^2) - (\sum p)(\sum p \ln q)}{n \sum p^2 - (\sum p)^2} = a'$$

$\therefore$ a = antilna'

Same way,

$$\ln b = \frac{n \sum (p \ln q) - (\sum p)(\sum \ln q)}{n \sum p^2 - (\sum p)^2} = b'$$

$\therefore$ b = antilnb'

So that, demand trend is Eq.(IV):

$$q = \text{antilna}'.(\text{antilnb}')^{p} \qquad \text{(IV)}$$

**Quantitative Example**

Functional relation between price and quantity is determined as $q = ab^{p}$. Quantities demanded at alternative prices have been listed below. Total cost function is established as TC = 200 + 2q + $0.01q^2$. The following year's price, as to an inflation approximately 40 %, is expected to be 25 m.u.

Calculation of the profits -before taxes- wanted.

| | | | | | | |
|---|---|---|---|---|---|---|
| p : | 5 | 7 | 8 | 10 | 13 | 18 |
| q : | 815 | 751 | 721 | 665 | 588 | 480 |

| | lnq | p | plnq | $p^2$ |
|---|---|---|---|---|
| | 6.70 | 5 | 33.50 | 25 |
| | 6.62 | 7 | 46.34 | 49 |
| | 6.58 | 8 | 52.64 | 64 |
| | 6.50 | 10 | 65.00 | 100 |
| | 6.38 | 13 | 82.94 | 169 |
| | 6.17 | 18 | 111.06 | 324 |
| $\sum$ s: | 38.95 | 61 | 391.48 | 731 |



$$\ln a = \frac{38{,}95(731) - 61(391{,}48)}{6(731) - 3721} = 6.9; \therefore a = 998$$

$$\ln b = \frac{6(391.48) - 61(38.95)}{665} = \frac{-27{,}07}{665} = -0.04; \ b = 0.96$$

Then,

$$q = 998(0.96)^{p}$$

Quantity for the following year is: $q = 998(0.96)^{25} = 360 \longrightarrow$ TR= 9000 m.u.

Total Cost, TC = 200 + 2(360) + 0.01($360^2$) = 2216

$\therefore$ Profits, K = 9000 — 2216 = 6784 m.u.

### 3. PROFITS ESTIMATION when DEMAND IS NORMALLY DISTRIBUTED

Total Cost, TC = $b_0 + b_1 q + b_2 q^2$ (I)

where,

$b_0$ = total fix costs (free from production)

q = quantity produced

$b_1 q + b_2 q^2$ = TVC (total variable costs)

letting unit variable cost = v = $b_1 + b_2 q$ , total cost becomes:

$$TC = b_0 + qv$$

After this short introduction, various states of profits will be determined according to the following fixations:

p = 15000 m.u.; $b_0 = 6.10^7$; v = 9000; i.e., the firm compensated all the costs of inputs and its overheads, if it can sell $10^4$ units (viz. 150 million m.u. of sales).

In addition, according to the estimations of the sales manager, sales expectation, E(q), i.e., average number of units sold, $\mu$ = 12000 units and the probability of sales being 6000 u. either side of this is approximately 95%, i.e.

$P(6000 < q < 18000) \doteq 0.95 \longrightarrow \sigma_q \doteq 3000$ u.

and demand (sales, here) is normally distributed:

$N(\mu = 12000;\ \sigma = 3000)$

as shown in Fig.1.

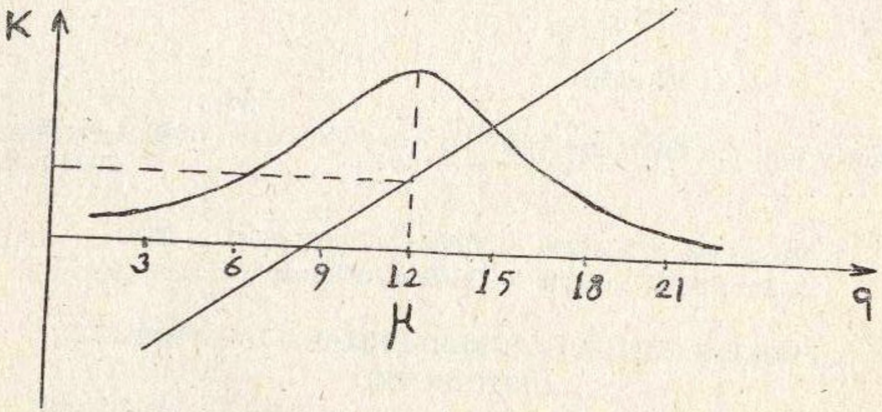



Fig. 1

Expected profits, $E(K) = E(q) \cdot (p\text{-}v) - b_0$

$= 12000(15000\text{—}9000) - 60000000 = 12.000.000$ m.u.

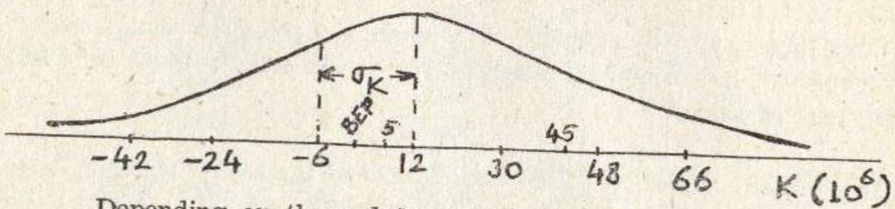
Standard deviation of profits:

$\sigma_K = \sigma_q (p\text{-}v) = 3000(6000) = 18.000.000$

So, probability distribution of profits is also known (as $10^6$ m.u.):

$N(\mu_K = 12;\ \sigma_K = 18)$

as sketched in Fig. 2 below.




Depending on these data and findings,

1. Probability of suffering no loss
2. Probability of profits' exceeding $10^7$ m.u.



3. Probability of suffering loss
4. Pfobability of suffering loss exceeding $15.10^6$ m.u.
5. Probability of profits' being 5 up to 45 million mu.u., have been estimated below.

1. $P(K \geqslant 0) = 0.5 + P[0<K<E(K)] = 0.05 + P\left[\frac{0\text{—}12}{18} < Z < 0\right]$

$\quad = 0.5 + 0.2475 \doteq 75\%$

2. $P(K>10^7) = 0.5 + P[10<K<E(K)] = 0.5 + P[-1/9 < Z \leqslant 0]$

$\quad = 05 + 0.0442 = 0.5442$

3. $P(K < 0) = 1 - 75\% = 25\%$

4. $P(K < -15.10^6) = 0.5 - P[-15 < K < E(K)]$

$\quad = 0.5 - P(-1.5 < Z < 0)$

$\quad = 0.5 - 0.4332 = 0.0668 \doteq 67\%0 = 0.067$

5. $P(5<K<45) = P[5 < K < E(K)] + P[E(K) < K < 45]$

$\quad = P(-7/18 < Z < 0) + P(0 < Z < 33/18)$

$\quad = 0.1506 + 0.4666 \doteq 62\%$

Up to this point, price, unit variable cost, overhead had been accepted unchanged; if they were also variable and their standard deviations were, successively, $\sigma_p = 1200$; $\sigma_v = 900$; $\sigma_{b_0} = 3(10^6)$, then standard deviation of profits, subjected to combined effect of these fastors, will also change and be calculated by Eq.(2):

$$\sigma_K = \sqrt{(\sigma_p^2 + \sigma_v^2)\left[\sigma_q^2 + E^2(q)\right] + \sigma_q^2\left[E(p) - E(v)\right]^2 + \sigma_{b_0}^2} \qquad (2)$$

When the data are put into Eq.(2), new standard deviation of profits is found as: $\sigma_K = 18.309.833$

Namely, standard error of profits has increased by 309833. The two situations are compared in Tablo 5.:

Table 5. Comparison of Expected Profits, Profits Standard Deviations, and Probabilistic Measures

| | Case I | Case II |
|---|---|---|
| Expected profits ($10^6$m.u.) | 12 | 12 |
| Standard Error of Profits | 18 | 18,309833 |
| **Probabilities :** | | |
| 1. Not to lose | 75% | 74% |
| 2. Profits exceedin $10^7$ | 54,4% | 54,3% |
| 3. Loss | 25% | 26% |
| 4. Loss over $15.10^6$ | 0.0668 | 0.0700 |
| 5. Profits 5 to 45 million m.u. | 0.62 | 0.6076 |

**REFERENCES**

1— YÖNEYLEM ARAŞTIRMASI (O.R), Ege Üniv. Press, 1979, İzmir. For inventory policies for frofit maximizing and courses of action fitting some other probability distributions, p.194 to 228. Prof. Dr. S. Akalın.

2— YÖNETİM EKONOMİSİ, Ticaret Matbaacılık, 1970, Prof. Dr. Sedat Akalın.

## Ö Z E T

Firmaların faaliyetlerini sürdürebilmeleri, bütün giderlerini aşacak tutarda satış yapmalarına bağlı bulunduğu herkesin malûmudur.

Gelecek kesinlikle belli olmadığından, satışların bir miktar hata payı ile tahmini işletme kararlarında ışık tutmaktadır. Satış miktarlarının tahmininden sonra, aynı gelecek zaman periyodu için fiyatların tahmini yapılıp tutarlar hesaplanabilmekte ve yine aynı döneme ait tahminî toplam maliyetleri gelirden çıkarmak suretiyle kâr tahmin olunmaktadır.

Satış tahminlerinin, pazar araştırması sonucu tesbit olunan talep fonksiyonlarına dayandırılma zorunluluğu bulunduğundan; talebin üstel dağılımlı, normal dağılımlı, eğrisel dağılımlı, logaritmik dağılımlı, poisson dağılımlı tekdüzen dağılıml, süreksiz dağılımlı vb. olduğu durumlar içinden, bu yazıda sadece ilk üç yâni, 1) talebin eğrisel trendli, 2) üstel fonksiyonlu, 3) normal dağılımlı olduğu durumlarda satış ve kâr tahminine değinilmiştir.